Math 53: Multivariable Calculus **Sections 212, 213**

Worksheet for 2020-08-31

Questions marked with ** are less relevant to the core material and/or more difficult.

Problem 1. A particle starts at the origin at time $t = 0$ and follows the path $x = f(t)$, $y = g(t)$ illustrated in Figure 1. At time $t = 1$, it returns to the origin. Compute

$$\int_0^1 f(t)g'(t)\,\mathrm{d}t \text{ and } \int_0^1 g(t)f'(t)\,\mathrm{d}t$$

by interpreting them in terms of areas. How are the two values related to each other?

**It turns out that this relationship between these two integrals holds *as long as the path ends where it starts* (as it does in this example). Can you explain why?

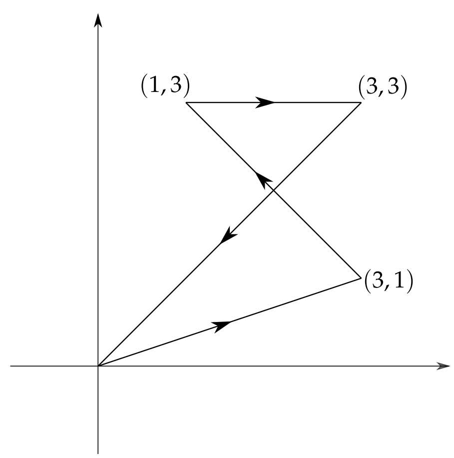

FIGURE 1. Problem 1

Problem 2. Find a Cartesian equation for the parametric curve $x = t^3 + t$, $y = t^2 + 2$. Then compute $\mathrm{d}y/\mathrm{d}x$, using (a) methods from Chapter 10, and (b**) implicit differentiation (hopefully at least one person in your group remembers how to do this!). Do you get the same answer?

Problem 3. There are two points on the curve

$$x = 2t^2, y = t - t^2, -\infty < t < \infty$$

where the tangent line passes through the point $(10, -2)$. Find these two points.

Problem 4.

(a) Write down a parametrization $x = f(t), y = g(t)$ for the circle $x^2 + y^2 = 1$ which starts at $(1, 0)$ when $t = 0$, goes around once counterclockwise, and ends at $(1, 0)$ when $t = 2\pi$.

(b) Adjust your parametrization so that it completes the revolution in 1 unit of time, instead of 2π (i.e. it returns to $(1, 0)$ at $t = 1$ instead of $t = 2\pi$). Note that you are still parametrizing $x^2 + y^2 = 1$; the circle itself has not changed.

(c) Take this curve (the unit circle centered at the origin) and stretch it horizontally by a factor of 3 (so you get an ellipse centered at the origin). Write down a Cartesian equation and a parametrization for the result.

(d) Take the ellipse from (c) and translate it up in the positive y-direction by 5 units (so you get an ellipse centered at $(0, 5)$). Write down a Cartesian equation and a parametrization for the result.

Problem 5** (Stereographic projection from the "north pole"). In Figure 2, the circle $x^2 + y^2 = 1$ has been depicted, together with a line passing through the points $(0, 1)$ and $(t, 0)$. This line intersects the circle at a point (other than $(0, 1)$), whose coordinates depend on the value of t. Find these coordinates $(f(t), g(t))$. Does the parametrization $x = f(t)$, $y = g(t)$, $-\infty < t < \infty$ trace out the entire unit circle?

See also: Stewart 10.1.40-44, which are similar in flavor (producing a parametrization from a geometric construction).

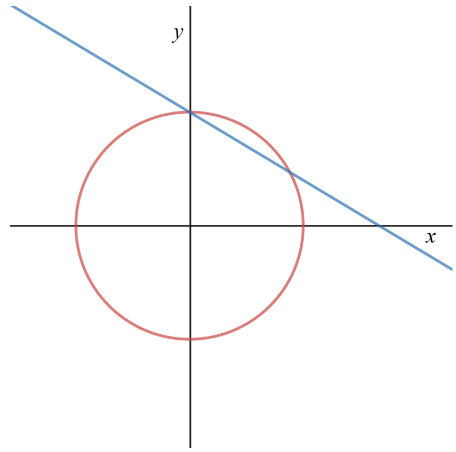

FIGURE 2. Problem 5